



1.6.8.4 Multiplication of complex numbers

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Introduction

- In this topic, we will
 - Define the multiplication of complex numbers
 - Focus on the polar representation
 - Look at a geometric interpretation
 - Make some observations regarding properties
 - Prove multiplication distributes over addition





Review

- Recall from secondary school the following two products:

$$(3.2 + 4.8x)(-5.1 + 9.6x) = -16.32 + 30.72x - 24.48x + 46.08x^2$$
$$= -16.32 + 6.24x + 46.08x^2$$

$$(0.5 - 4.9\sqrt{2})(7.3 - 1.8\sqrt{2}) = 3.65 - 0.9\sqrt{2} - 35.77\sqrt{2} + 8.82\sqrt{2}^2$$
$$= 21.29 - 36.67\sqrt{2}$$

- You used the FOIL rule:

First Outside Inside Last





Complex multiplication

- Recall that $j \stackrel{\text{def}}{=} \sqrt{-1}$, thus if

$$z = \alpha + \beta j$$

$$w = \gamma + \delta j$$

it follows that $zw = (\alpha + \beta j)(\gamma + \delta j)$

$$= \alpha\gamma + \alpha\delta j + \beta\gamma j + \beta\delta j^2$$

$$= \alpha\gamma + (\alpha\delta + \beta\gamma)j + \beta\delta(-1)$$

$$= (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j$$

$$\text{but } j^2 = \sqrt{-1}^2 = -1$$





Complex multiplication

- Some examples:

$$\begin{aligned}(3 + 4j)(2 + 5j) &= 3 \cdot 2 + 3 \cdot 5j + 4 \cdot 2j + 4 \cdot 5j^2 \\ &= (6 - 20) + (15 + 8)j \\ &= -14 + 23j\end{aligned}$$

$$\begin{aligned}(1 + 2j)(-2 + j) &= 1 \cdot (-2) + j + 2 \cdot (-2)j + 2j^2 \\ &= (-2 - 2) + (1 - 4)j \\ &= -4 - 3j\end{aligned}$$





Complex multiplication

- Some more examples:

$$\begin{aligned}(3.2 + 4.8j)(-5.1 + 9.6j) &= 3.2 \cdot (-5.1) + 3.2 \cdot 9.6j + 4.8 \cdot (-5.1)j - 4.8 \cdot 9.6 \\ &= (-16.32 - 46.08) + (30.72 - 24.48)j \\ &= -62.4 + 6.24j\end{aligned}$$

$$\begin{aligned}(0.5 - 4.9j)(7.3 - 1.8j) &= 0.5 \cdot 7.3 + 0.5 \cdot (-1.8)j + (-4.9) \cdot 7.3j - (-4.9) \cdot (-1.8) \\ &= (3.65 - 8.82) + (-0.9 - 35.77)j \\ &= -5.17 - 36.67j\end{aligned}$$





Polar representation

- Problem: there is no reasonable geometric interpretation of complex multiplication using the rectangular representation...

$$zw = (\alpha + \beta j)(\gamma + \delta j) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j$$

- Instead, we will have to go to the polar representation





Polar representation

- Let $z = |z| \angle \theta$ and $w = |w| \angle \phi$
 - Question: What is zw ?
 - We only have the definition in rectangular representation, so

$$\begin{aligned} zw &= (|z| \angle \theta)(|w| \angle \phi) \\ &= (|z| \cos(\theta) + |z| \sin(\theta) j)(|w| \cos(\phi) + |w| \sin(\phi) j) \\ &= |z||w| \cos(\theta) \cos(\phi) - |z||w| \sin(\theta) \sin(\phi) \\ &\quad + |z||w| \cos(\theta) \sin(\phi) j + |z||w| \sin(\theta) \cos(\phi) j \\ &= |z||w| (\cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi)) \\ &\quad + |z||w| (\cos(\theta) \sin(\phi) + \sin(\theta) \cos(\phi)) j \end{aligned}$$





Polar representation

$$\begin{aligned}zw &= |z||w|(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) \\ &\quad + |z||w|(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi))j\end{aligned}$$

- Recall your double-angle formulas:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\sin(\theta + \phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)$$

- Thus,

$$\begin{aligned}zw &= |z||w|\cos(\theta + \phi) + |z||w|\sin(\theta + \phi)j \\ &= |z||w|\angle(\theta + \phi)\end{aligned}$$





Polar representation

- Thus, $zw = (|z| \angle \theta)(|w| \angle \phi) = |z||w| \angle (\theta + \phi)$
 - Multiply the magnitudes and add the angles

- Examples:

- The value $j = 1 \angle \frac{\pi}{2}$ and

$$j^2 = \left(1 \angle \frac{\pi}{2}\right)\left(1 \angle \frac{\pi}{2}\right) = 1^2 \angle \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1 \angle \pi = -1$$

- Given $1 + j = \sqrt{2} \angle \frac{\pi}{4}$ and $-3 = 3 \angle \pi$,

we have $(1 + j)(-3) = -3 - 3j$

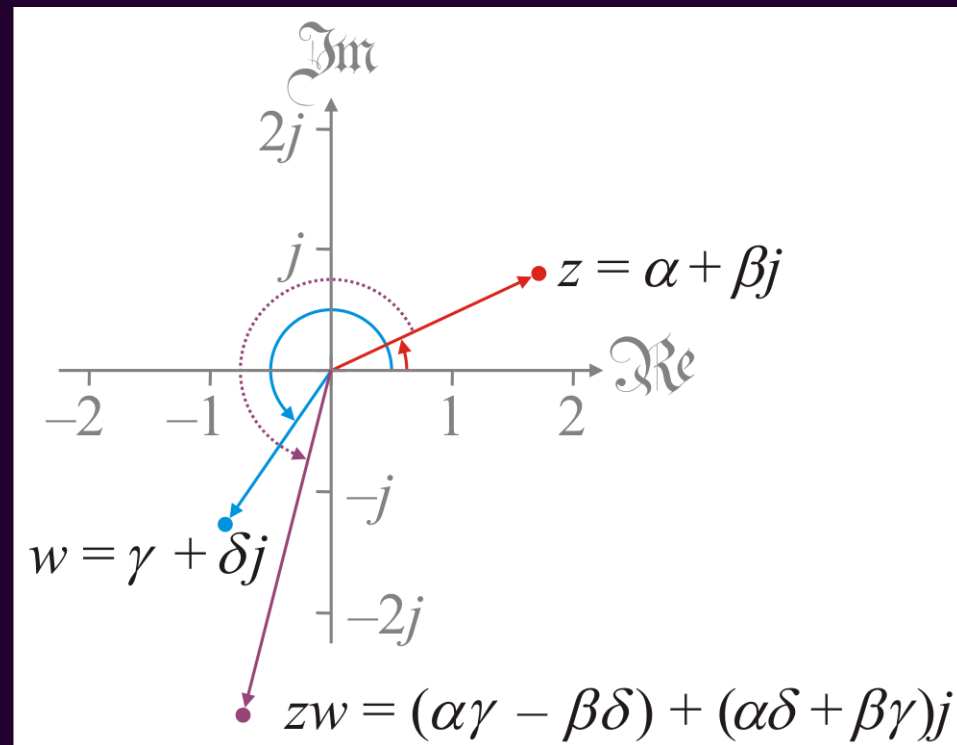
which equals $\left(\sqrt{2} \angle \frac{\pi}{4}\right)(3 \angle \pi) = \left(3\sqrt{2} \angle \frac{5\pi}{4}\right)$





Geometric interpretation

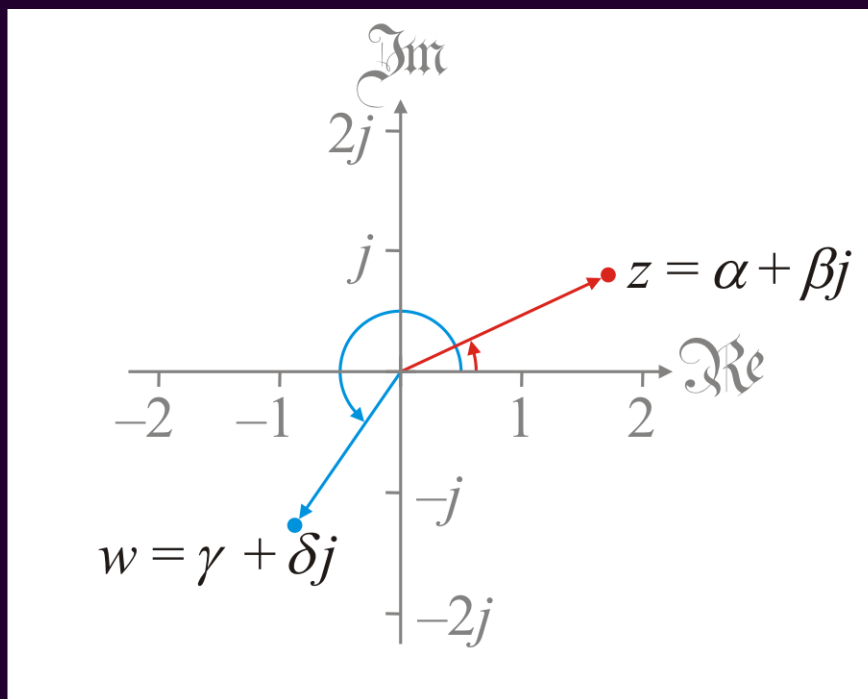
- If we have z and w , their product zw has:
 - The sum of the angles
 - The product of the absolute values





Geometric interpretation

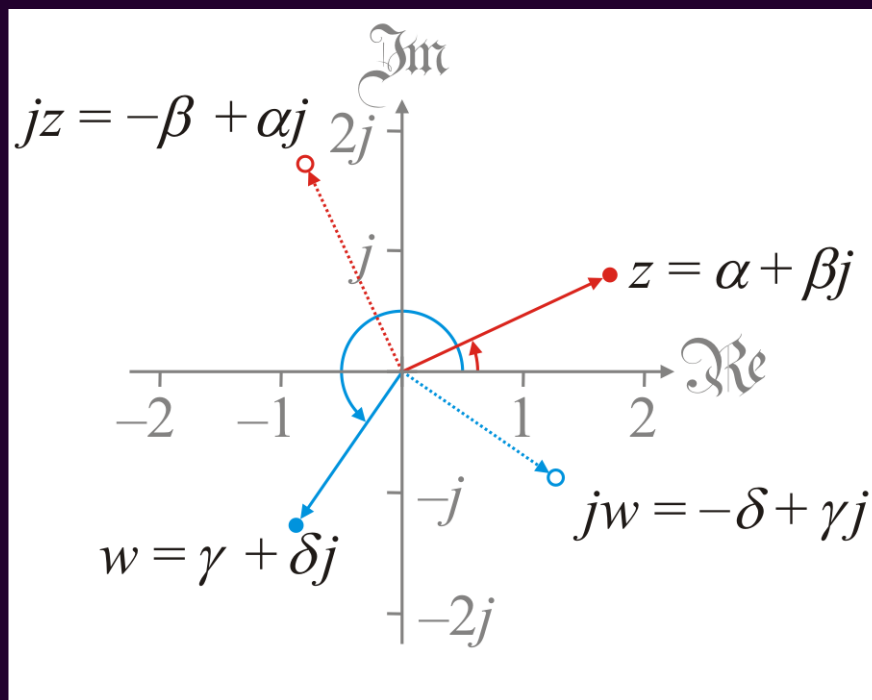
- Let's consider a few products in particular
 - Here are z and w





Geometric interpretation

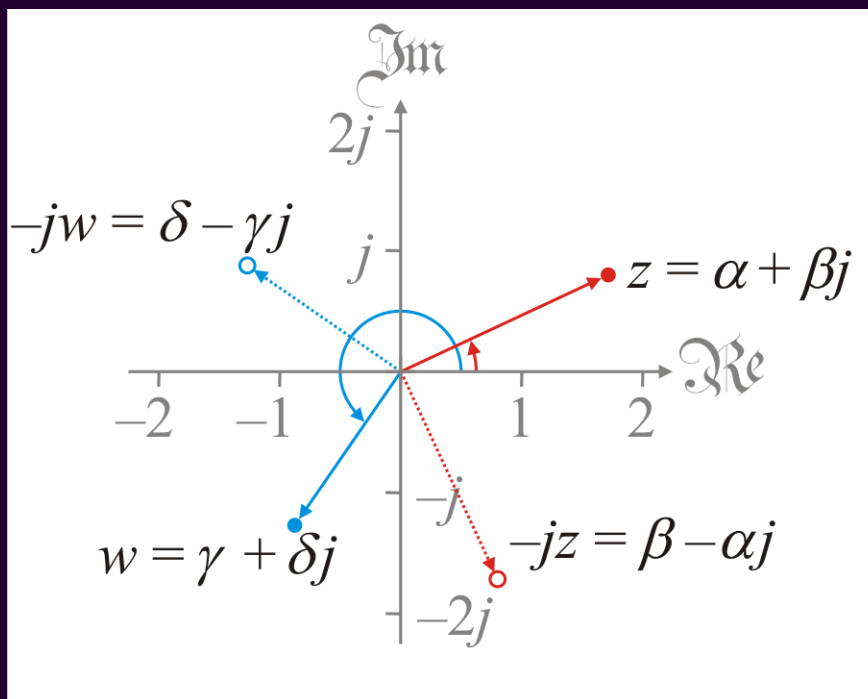
- Let's consider a few products in particular
 - Recall that $j = 1 \angle \frac{\pi}{2}$
 - Multiplying by j rotates complex numbers by 90° clockwise





Geometric interpretation

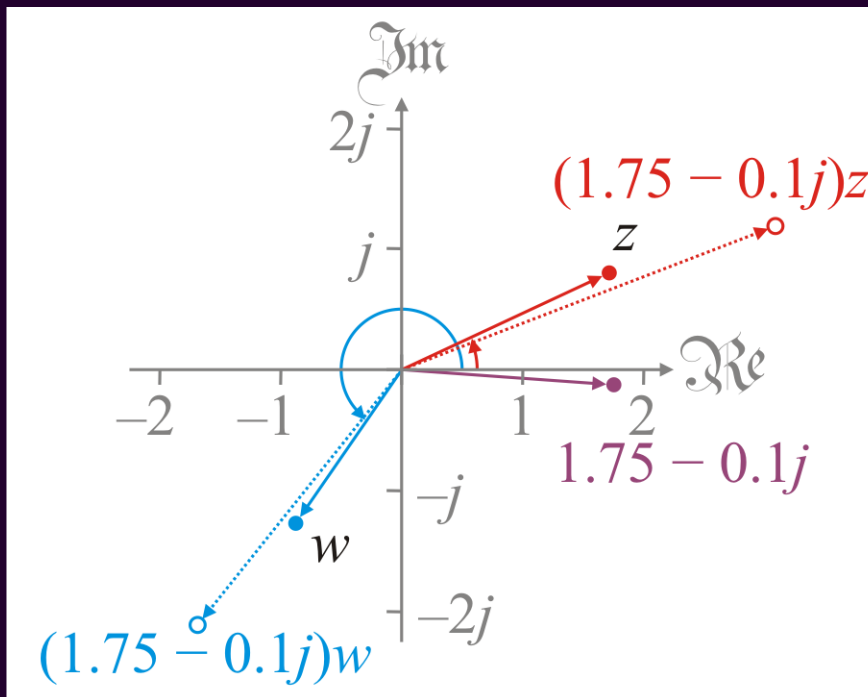
- Let's consider a few products in particular
 - Recall that $-j = 1 \angle -\frac{\pi}{2} = 1 \angle \frac{3\pi}{2}$
 - Multiplying by $-j$ rotates by 90° counter-clockwise





Geometric interpretation

- Let's consider a few products in particular
 - Suppose we are multiplying by $1.75 - 0.1j$
 - This will stretch the numbers by a little more than 75%
 - It will rotate the numbers a few degrees counter-clockwise





Absolute value

Theorem: $|zw| = |z||w|$

The simple proof:

Recall that for $z = |z| \angle \theta$ and $w = |w| \angle \phi$,
the product is $zw = |z||w| \angle (\theta + \phi)$
so $|zw| = |z||w|$. *QED*





Absolute value

$$zw = (\alpha + \beta j)(\gamma + \delta j) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j$$

Theorem: $|zw| = |z||w|$

The tedious proof:

$$\begin{aligned}|zw|^2 &= (\alpha\gamma - \beta\delta)^2 + (\alpha\delta + \beta\gamma)^2 \\&= \alpha^2\gamma^2 - \cancel{2\alpha\beta\gamma\delta} + \beta^2\delta^2 + \alpha^2\delta^2 + \cancel{2\alpha\beta\gamma\delta} + \beta^2\gamma^2 \\&= \alpha^2\gamma^2 + \beta^2\delta^2 + \alpha^2\delta^2 + \beta^2\gamma^2 \\&= \alpha^2(\gamma^2 + \delta^2) + \beta^2(\delta^2 + \gamma^2) \\&= (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) \\&= |z|^2 |w|^2\end{aligned}$$

Taking the square root of both sides:

$$|zw| = \sqrt{|zw|^2} = \sqrt{|z|^2 |w|^2} = \sqrt{|z|^2} \sqrt{|w|^2} = |z||w| \quad \text{QED}$$





Commutativity

Theorem: Complex multiplication is commutative.

The easy proof:

If $z = |z| \angle \theta$ and $w = |w| \angle \phi$, then

$$zw = |z||w| \angle (\theta + \phi)$$

But addition and multiplication of real numbers are commutative, so $zw = |z||w| \angle (\theta + \phi)$

$$= |w||z| \angle (\phi + \theta)$$

$$= wz \quad \text{QED}$$





Commutativity

Theorem: Complex multiplication is commutative.

The tedious proof:

If $z = \alpha + \beta j$ and $w = \gamma + \delta j$, then

$$zw = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j$$

But multiplication of real numbers is commutative, so

$$zw = (\gamma\alpha - \delta\beta) + (\delta\alpha + \gamma\beta)j$$

$$= \gamma\alpha + \delta\alpha j + \gamma\beta j + \delta\beta j^2$$

$$= (\gamma + \delta j)(\alpha + \beta j)$$

$$= wz \quad \text{QED}$$





Associativity

Theorem: Complex multiplication is associative.

The easy proof:

If $z_1 = |z_1| \angle \theta_1$, $z_2 = |z_2| \angle \theta_2$ and $z_3 = |z_3| \angle \theta_3$, then

$$z_1(z_2 z_3) = z_1(|z_2||z_3| \angle (\theta_2 + \theta_3))$$

$$= |z_1|(|z_2||z_3|) \angle (\theta_1 + (\theta_2 + \theta_3))$$

But addition and multiplication of real numbers are commutative, so $z_1(z_2 z_3) = |z_1|(|z_2||z_3|) \angle (\theta_1 + (\theta_2 + \theta_3))$

$$= (|z_1||z_2|)|z_3| \angle ((\theta_1 + \theta_2) + \theta_3)$$

$$= (|z_1||z_2| \angle (\theta_1 + \theta_2)) z_3$$

$$= (z_1 z_2) z_3 \quad \text{QED}$$





Associativity

Theorem: Complex multiplication is associative.

The tedious proof:

$$\begin{aligned} \text{If } z_1 &= \alpha + \beta j, z_2 = \gamma + \delta j \text{ and } z_3 = \varepsilon + \zeta j, \\ z_1(z_2 z_3) &= (\alpha + \beta j)((\gamma + \delta j)(\varepsilon + \zeta j)) \\ &= (\alpha + \beta j)((\gamma\varepsilon - \delta\zeta) + (\gamma\zeta + \delta\varepsilon)j) \\ &= (\alpha(\gamma\varepsilon - \delta\zeta) - \beta(\gamma\zeta + \delta\varepsilon)) + (\alpha(\gamma\zeta + \delta\varepsilon) - \beta(\gamma\varepsilon - \delta\zeta))j \\ &= (\alpha\gamma\varepsilon - \alpha\delta\zeta - \beta\gamma\zeta - \beta\delta\varepsilon) + (\alpha\gamma\zeta + \alpha\delta\varepsilon - \beta\gamma\varepsilon + \beta\delta\zeta)j \\ &= (\alpha\gamma\varepsilon - \beta\delta\varepsilon - \alpha\delta\zeta - \beta\gamma\zeta) + (\alpha\delta\varepsilon - \beta\gamma\varepsilon + \alpha\gamma\zeta + \beta\delta\zeta)j \\ &= ((\alpha\gamma - \beta\delta)\varepsilon - (\alpha\delta + \beta\gamma)\zeta) + ((\alpha\delta - \beta\gamma)\varepsilon + (\alpha\gamma + \beta\delta)\zeta)j \\ &= ((\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j)(\varepsilon + \zeta j) \\ &= (z_1 z_2) z_3 \quad \text{QED} \end{aligned}$$





The multiplicative identity

Theorem: $1 = 1 + 0j = 1 \angle 0$ is the multiplicative identity.

The easy proof:

$$\begin{aligned}\text{If } z = |z| \angle \theta, \text{ then } 1 \cdot z &= (1 \angle 0)(|z| \angle \theta) \\ &= 1 \cdot |z| \angle (0 + \theta) \\ &= |z| \angle \theta = z \quad \text{QED}\end{aligned}$$

The tedious proof:

$$\begin{aligned}\text{If } z = \alpha + \beta j, \text{ then } 1 \cdot z &= (1 + 0j)(\alpha + \beta j) \\ &= 1 \cdot \alpha + 1 \cdot \beta j + 0 \cdot \alpha j + 0 \cdot \beta \\ &= \alpha + \beta j = z \quad \text{QED}\end{aligned}$$





Observations

- Here are some more results from elementary school:
 - Multiplying two positive reals yields a positive real:
$$(|z_1| \angle 0)(|z_2| \angle 0) = |z_1||z_2| \angle (0 + 0) = |z_1||z_2| \angle 0$$
 - Multiplying two negative reals yields a positive real
 $(|z_1|$ [REDACTED]
 - Multiplying positive and negative real yields a negative real
$$(|z_1| \angle 0)(|z_2| \angle \pi) = |z_1||z_2| \angle (0 + \pi) = |z_1||z_2| \angle \pi$$





Multiplication distributes over addition

Theorem: Complex multiplication distributes over addition.

Proof: If $z_1 = \alpha + \beta j$, $z_2 = \gamma + \delta j$ and $z_3 = \varepsilon + \zeta j$,

$$\begin{aligned} z_1(z_2 + z_3) &= (\alpha + \beta j)((\gamma + \delta j) + (\varepsilon + \zeta j)) \\ &= (\alpha + \beta j)((\gamma + \varepsilon) + (\delta + \zeta)j) \\ &= (\alpha(\gamma + \varepsilon) - \beta(\delta + \zeta)) + (\beta(\gamma + \varepsilon) + \alpha(\delta + \zeta))j \end{aligned}$$

But real multiplication distributes over real addition:

$$\begin{aligned} &= (\alpha\gamma + \alpha\varepsilon - \beta\delta - \beta\zeta) + (\beta\gamma + \beta\varepsilon + \alpha\delta + \alpha\zeta)j \\ &= (\alpha\gamma - \beta\delta) + (\beta\gamma + \alpha\delta)j + (\alpha\varepsilon - \beta\zeta) + (\beta\varepsilon + \alpha\zeta)j \\ &= (\alpha + \beta j)(\gamma + \delta j) + (\alpha + \beta j)(\varepsilon + \zeta j) \\ &= z_1 z_2 + z_1 z_3 \quad \text{QED} \end{aligned}$$





Important takeaways

- The most important takeaways are
 - Multiplying complex numbers in the rectangular representation is no more than multiplying two linear polynomials
 - In the polar representation, it is the product of the magnitudes and the sum of the angles
 - Given n complex numbers being multiplied, you can multiply them in any order, and you'll get the same result
 - For example,

$$\begin{aligned} & (3.9 + 5.7j)(1.8 - 8.6j)(-0.5 + 3.4j)(1.9 - 3.1j) \\ & \quad (3.9 + 8.4j)(-6.4 + 1.5j)(2.2 + 4.2j)(4.1 - 7.0j) \\ & \text{equals } -550769.08882032 - 1687878.63288576j \end{aligned}$$





Summary

- In this topic, we've introduced complex multiplication
 - Very similar to polynomial multiplication
 - The geometric interpretation is reasonably easy
 - We saw that
 - It is commutative
 - It is associative
 - $1 + 0j$ is the multiplicative identity
 - Multiplication distributes over addition





References

- [1] https://en.wikipedia.org/wiki/Complex_number#Addition_and_subtraction
- [2] https://en.wikipedia.org/wiki/Absolute_value#Complex_numbers





Acknowledgments

Tina Hanna for correcting a typo.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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